Polyhedral Finite Elements

MSc Computational Mechanics: Software Lab Project

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Abstract

The purpose of this poster is to portray the development of polyhedral finite element formulations, by calculating the shape functions numerically for any point. These element formulations are expected to give better results in areas of high gradients. Reliability is tested by implementing elements for heat transfer problems and by comparing the results with analytical solutions.

Introduction

Many types of elements have been developed for a wide range of applications. The standard elements are triangles or quadrilaterals but with the increase in the complexity of engineering problems arbitrary polyhedral elements come into light. We believe these elements could offer two benefits. A coarse multi-noded mesh could perform as accurately as a fine regular quad mesh and it also offers a higher flexibility for mesh generation and allows better mapping of geometry. The main applications for these type of polyhedral elements are in complex fluid flow or heat transfer problems.

Interpolation functions at certain points of these elements are developed by using the Voronoi decomposition, which is a simple mathematical object which determines the nearest neighbour decomposition for a set of points in Euclidean space. The new element formulations are included in a non commercial FEM code and tested for heat transfer problems. The numerical results obtained from FEM are compared with analytical solution.

Method

The Method to derive the shape functions makes use of the Voronoi decomposition in order to obtain the influence areas of certain points inside each element.

Voronoi Decomposition

The Voronoi decomposition is a simple mathematical procedure that determines the nearest neighbours for a set of points in Euclidean space. The decomposition is done by drawing the perpendicular bisectors of each neighbouring point set. For our purpose this actually has to be done twice to derive the shape functions for arbitrary polyhedral elements.

Voronoi Region of First Order

The first order decomposition is the initial decomposition of the polyhedral which shows the influence area of each corner node of the element. This is done by drawing the perpendicular bisector of two neighbouring nodes as shown in figure 1. In this figure the first order Voronoi decomposition is displayed for a regular hexagon.

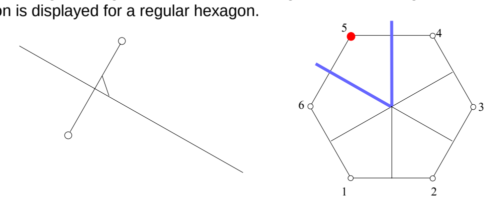


Figure 1: Voronoi decomposition of two nodes and decomposition of first order

The area bound by the blue lines indicate the influence area of corner node 5. This influence area represents the area in which all points are closer (or equally close) to node 5 than they are to any other node of the element. This is the actual influence area for a nodal displacement applied at node number 5, when it comes to finite element approximation.

Voronoi Region of Second Order

The second order decomposition results in the influence area of an arbitrary point *P* inside the element. For this purpose the perpendicular bisectors between each corner node and point *P* are drawn. As shown in figure 2a these bisectors will define the influence area of point *P*. The influence area shows the set of points which are closer to point *P* than to any corner node. Overlaying the Voronoi decomposition of first and second order, results in figure 2b. The pink triangle marks the area of influence that the value of node *i* has on the value of point *P*. Taking the influence area of point *P* with regard to each corner node times the nodal displacement at each corner node, will give the Displacement at node *P*.

In order to find the shape function value *i* at point *P* the pink area is divided by the whole influence area (area inside the blue lines) of point *P*. As can be seen the sum of all shape function values at point *P* will be one.

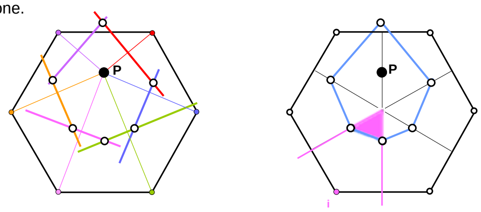


Figure 2: **a**. Finding the voronoi decomposition of second order.
b. First and Second order voronoi regions.

Gauss point integration

In order to derive the element formulations for different polyhedral elements, it is necessary to calculate the influence of each corner node of the polygon to its gauss points. Six gauss points are used to integrate over the polygon's area. The derivatives in x and y direction at each gauss point is derived by using the central difference method.

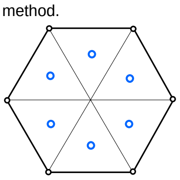
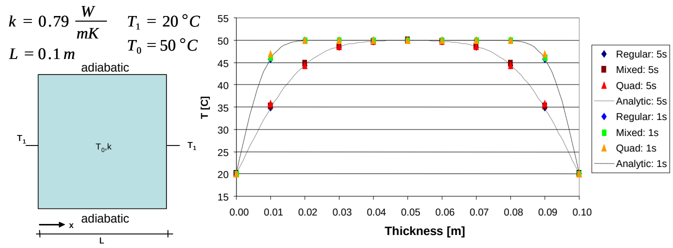


Figure 3: Gauss point distribution for Hexahedral Element.

Results

1D homogeneous wall cooling with time

1D transient heat conduction problem at time = 5 sec. This example provides a high temperature gradient near the left and right edge. We see that all meshes are accurate.



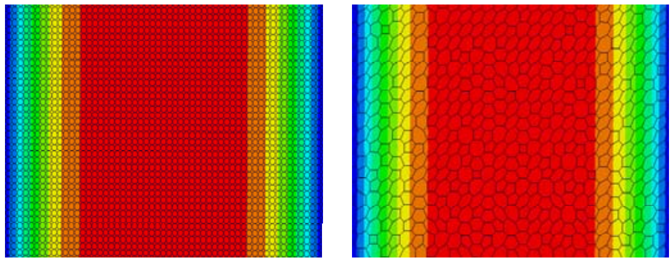
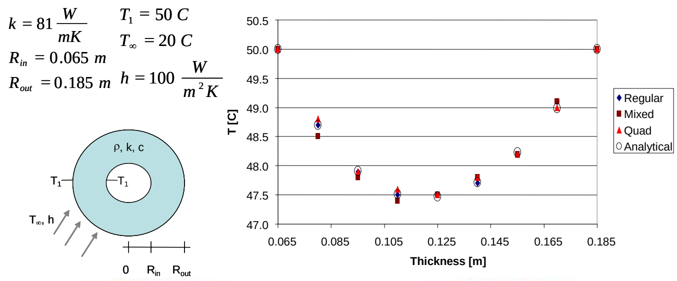


Figure 4: Results at t = 5 sec

2D conduction in a disk (bound. cond.: temp/temp/conv.)

We analyze a 2D, steady state heat conduction in a disk with temperature boundary condition on the inner and outer surface. The temperature distribution is plotted along the radius. We can see that the regular mesh provides the most accurate results. We attribute this to the meshes fine scale over the other two meshes.



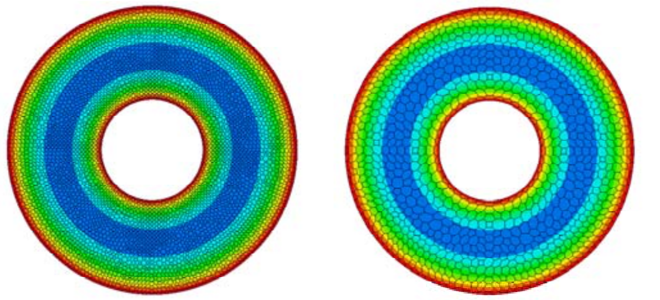


Figure 5: Results

Conclusion

In conclusion, through this project we developed and implemented the element formulations for 5-10 noded polyhedral elements in a finite element program and checked their validity on heat transfer problems against analytic solutions. From the results, we conclude that the elements and the finite element code are working properly.

To asses the concrete benefits of the elements, more testing is needed in high gradient regions and further testing with respect to mesh coarseness should be carried out to assess whether a coarse multi-noded mesh performs as accurately as a fine quad mesh.

References

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